E. H. Avrett

QUESTIONS OF CONSISTENCY AND CONVERGENCE IN THE SOLUTION OF MULTILEVEL TRANSFER PROBLEMS

Questions of Consistency and Convergence in the Solution of Multilevel Transfer Problems

Eugene H. Avrett
Smithsonian Astrophysical Observatory and Harvard College Observatory
Cambridge, Massachusetts

ABSTRACT

In the solution of line transfer problems it is necessary to deal with ratios of atomic populations. For a three-level problem with three radiative transitions, we must calculate the ratios n_2/n_1, n_3/n_1, and n_3/n_2 by means of separate equations. Solutions obtained by certain iterative methods are sometimes inconsistent, in the sense that $(n_3/n_1)/(n_2/n_1) \neq n_3/n_2$. Similar difficulties also arise in the solution of multiplet problems, where two or more pairs of radiative transitions connect with two fine-structure levels j and i. The source functions determined for each pair of transitions must not imply different values of n_j/n_i. We trace the difficulty to the calculation of net radiative rates at small optical depths. Equations appropriate at large optical depths are not always appropriate in the region $\tau \lesssim 1$. The problems of consistency and convergence are not independent; inconsistencies often occur in cases that have poor convergence properties. We describe a general computational procedure, having good convergence properties, which ensures that a consistent iterative solution is obtained.

1. Introduction

In the spring of 1967, Rudolf Loeser and the author developed a general computer program, called PANDORA, for the solution of multilevel line transfer problems. It was found that solutions could be obtained without difficulty in most of the cases that were attempted. In certain cases, however, the computed solutions exhibited a basic inconsistency. A year was spent on further development before the difficulty was understood and corrected. The purpose of this paper is to spare other investigators similar delay.

Using PANDORA in its original form, we readily obtained solutions for the atomic configurations shown in Figure 1; the broken lines represent assumed collisional transitions, and the solid lines represent both radiative and collisional transitions. These cases are characterized by the absence of any complete circuit of radiative transitions.

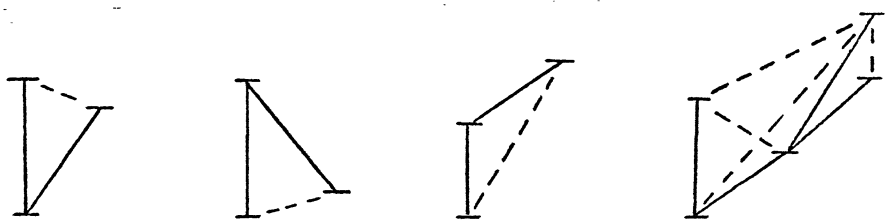
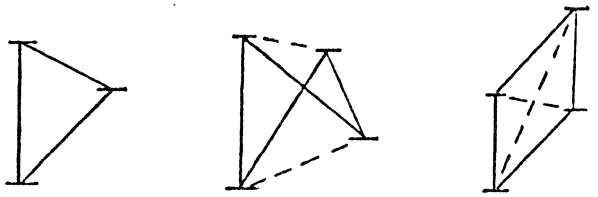


Figure 1

For the configurations shown in Figure 2, however, the program sometimes gave solutions that exhibited an inconsistency. In Case a the population ratio of the two upper levels established by the two lower radiative transitions differed from the population ratio determined by the upper transition. In the two other cases, b and c, the population ratio n_j/n_i established by one pair of radiative transitions differed from the ratio established by the other pair.



(a) (b) (c)

Figure 2

Such inconsistencies are normally encountered during the initial stages of an iterative solution, but the procedure we use has good convergence properties in the sense that a stable solution is usually obtained after about five iterations. We find that when an inconsistency arises the stable solution is not always independent of the starting solution. In some cases, however, this stable solution is completely unaffected by further iteration.

Not only does the inconsistency arise when we deal with only certain atomic configurations, but it is also sometimes removed when we choose different values for certain atmospheric parameters. This point is illustrated later.

2. Outline and Conclusions

In Section 3 we derive the basic equations that relate a given line source function to the net radiative rates for other line transitions. The source function S_{ji} depends on each net radiative bracket ρ_{mn}, $mn \neq ji$, and each ρ can be determined from the values of S. The basic iterative procedure consists of a successive evaluation of these functions.

In Section 4 we describe the usual type of iterative procedure, which involves a standard technique for the evaluation of ρ. An alternative method for the evaluation of ρ is discussed in Section 5. Our modified procedure is described in Section 6. We remark on the use of single radiative rates and total net rates in Sections 7 and 8.

Numerical results are presented in Section 9. On the basis of these results we conclude the following: 1) Except in certain circumstances, the unmodified procedure described in Section 4 has good convergence properties and leads to a consistent solution. 2) Difficulties with this procedure arise in the case of a) depth variations in certain rate coefficients at small optical depths, and b) strong coupling between similar radiative transitions. 3) The modified procedure we describe in Section 6 leads to a consistent solution in all cases, and has good convergence properties.

3. Equations for a Three-Level Atom

We can describe the basic features of our method of solution by using the general three-level case as an example. The statistical equilibrium equations for levels 1, 2, and 3 are

$$\begin{aligned} n_1(P_{12} + P_{13} + P_{1k}) &= n_2 P_{21} + n_3 P_{31} + n_k P_{k1} \quad , \\ n_2(P_{21} + P_{23} + P_{2k}) &= n_1 P_{12} + n_3 P_{32} + n_k P_{k2} \quad , \\ n_3(P_{31} + P_{32} + P_{3k}) &= n_1 P_{13} + n_2 P_{23} + n_k P_{k3} \quad . \end{aligned} \tag{1}$$

We eliminate n_k using the continuum equation

$$n_k(P_{k1} + P_{k2} + P_{k3}) = n_1 P_{1k} + n_2 P_{2k} + n_3 P_{3k} \quad , \tag{2}$$

which is the sum of equations (1), and obtain

$$\begin{aligned} n_1(P_{12} + \mathcal{P}_{12} + P_{13} + \mathcal{P}_{13}) &= n_2(P_{21} + \mathcal{P}_{21}) + n_3(P_{31} + \mathcal{P}_{31}) \quad , \\ n_2(P_{21} + \mathcal{P}_{21} + P_{23} + \mathcal{P}_{23}) &= n_1(P_{12} + \mathcal{P}_{12}) + n_3(P_{32} + \mathcal{P}_{32}) \quad , \\ n_3(P_{31} + \mathcal{P}_{31} + P_{32} + \mathcal{P}_{32}) &= n_1(P_{13} + \mathcal{P}_{13}) + n_2(P_{23} + \mathcal{P}_{23}) \quad , \end{aligned} \tag{3}$$

where

$$\mathcal{P}_{ij} = P_{ik}P_{kj}/(P_{k1} + P_{k2} + P_{k3}) \quad . \tag{4}$$

Any two of the above three equations can be used to calculate n_2/n_1 and n_3/n_1. The individual number densities can be obtained from these two ratios as follows. The total number density

$$n = n_1 + n_2 + n_3 + n_k \tag{5}$$

is presumed known. From this equation we write

$$n_1 = \frac{n}{1 + \frac{n_2}{n_1} + \frac{n_3}{n_1} + \frac{n_k}{n_1}} \quad , \tag{6}$$

and it follows from equation (2) that

$$\frac{n_k}{n_1} = \frac{P_{1k} + \frac{n_2}{n_1} P_{2k} + \frac{n_3}{n_1} P_{3k}}{P_{k1} + P_{k2} + P_{k3}} \quad . \tag{7}$$

The values of n_2 and n_3 can be found from n_1, n_2/n_1, and n_3/n_1.

The bound-bound rate coefficients are given by

$$\left.\begin{aligned} P_{ji} &= A_{ji} + \mathcal{B}_{ji}\bar{J}_{ji} + C_{ji} \quad , \\ P_{ij} &= \mathcal{B}_{ij}\bar{J}_{ji} + C_{ij} \quad . \end{aligned}\right\} \quad j > i \tag{8}$$

The bound-bound radiative rates are expressed as net rates:

$$n_j(A_{ji} + \mathcal{B}_{ji}\bar{J}_{ji}) - n_i\mathcal{B}_{ij}\bar{J}_{ji} = n_jA_{ji}\rho_{ji} \quad , \quad j > i \quad , \tag{9}$$

$$\rho_{ji} = 1 - \frac{\bar{J}_{ji}}{S_{ji}} \quad , \tag{10}$$

$$S_{ji} = \frac{2h\nu_{ji}^3/c^2}{\frac{\omega_j}{\omega_i}\frac{n_i}{n_j} - 1} \quad ; \tag{11}$$

S_{ji} is the line source function, and ρ_{ji} the net radiative bracket. The three statistical equilibrium equations then become

$$\begin{aligned} n_1(Z_{12} + Z_{13}) &= n_2(A_{21}\rho_{21} + Z_{21}) + n_3(A_{31}\rho_{31} + Z_{31}) \quad , \\ n_2(A_{21}\rho_{21} + Z_{21} + Z_{23}) &= n_1Z_{12} + n_3(A_{32}\rho_{32} + Z_{32}) \quad , \\ n_3(A_{31}\rho_{31} + Z_{31} + A_{32}\rho_{32} + Z_{32}) &= n_1Z_{13} + n_2Z_{23} \quad , \end{aligned} \tag{12}$$

where

$$Z_{ij} = C_{ij} + \mathcal{P}_{ij} \quad . \tag{13}$$

To deal with the 2-1 transition, we write the second- and third-level equations as

$$\begin{aligned} \frac{n_2}{n_1}(A_{21}\rho_{21} + Z_{21} + Z_{23}) - \frac{n_3}{n_1}(A_{32}\rho_{32} + Z_{32}) &= Z_{12} \quad , \\ -\frac{n_2}{n_1}Z_{23} + \frac{n_3}{n_1}(A_{31}\rho_{31} + Z_{31} + A_{32}\rho_{32} + Z_{32}) &= Z_{13} \quad . \end{aligned} \tag{14}$$

We solve this pair of equations for n_2/n_1 and substitute the result on the right-hand-side of the expression

$$S_{21} = \frac{2h\nu_{21}^3/c^2}{\frac{\omega_2}{\omega_1}\frac{n_1}{n_2} - 1} \quad . \tag{15}$$

Then

$$S_{21} = \frac{\bar{J}_{21} + \epsilon_{2,21}(1 - \beta_{21})B_{21}}{1 + \epsilon_{1,21} - \beta_{21}\epsilon_{2,21}} \quad , \tag{16}$$

where B_{21} is the Planck function, $\beta_{21} = \exp(-h\nu_{21}/kT)$, and where

$$\epsilon_{1,21} = \frac{1}{A_{21}}\left[Z_{21} + \frac{(A_{31}\rho_{31} + Z_{31})Z_{23}}{A_{31}\rho_{31} + Z_{31} + A_{32}\rho_{32} + Z_{32}}\right] \quad , \tag{17}$$

and

$$\epsilon_{2,21} = \frac{\omega_1}{\omega_2\beta_{21}}\frac{1}{A_{21}}\left[Z_{12} + \frac{Z_{13}(A_{32}\rho_{32} + Z_{32})}{A_{31}\rho_{31} + Z_{31} + A_{32}\rho_{32} + Z_{32}}\right] \quad . \tag{18}$$

For the 3-1 transition we follow the same procedure starting with the third- and second-level equations. Then

$$S_{31} = \frac{\bar{J}_{31} + \epsilon_{2,31}(1 - \beta_{31})B_{31}}{1 + \epsilon_{1,31} - \beta_{31}\epsilon_{2,31}} \quad , \tag{19}$$

where

$$\epsilon_{1,31} = \frac{1}{A_{31}}\left[Z_{31} + \frac{(A_{32}\rho_{32} + Z_{32})(A_{21}\rho_{21} + Z_{21})}{Z_{23} + A_{21}\rho_{21} + Z_{21}}\right] \quad , \tag{20}$$

and

$$\epsilon_{2,31} = \frac{\varpi_1}{\varpi_3 \beta_{31}} \frac{1}{A_{31}} \left[Z_{13} + \frac{Z_{23} Z_{12}}{Z_{23} + A_{21} \rho_{21} + Z_{21}} \right] . \tag{21}$$

For the 3-2 transition the third and first level equations are used to obtain

$$S_{32} = \frac{\bar{J}_{32} + \epsilon_{2,32}(1 - \beta_{32}) B_{32}}{1 + \epsilon_{1,32} - \beta_{32} \epsilon_{2,32}} , \tag{22}$$

where

$$\epsilon_{1,32} = \frac{1}{A_{32}} \left[Z_{32} + \frac{(A_{31} \rho_{31} + Z_{31}) Z_{12}}{Z_{13} + Z_{12}} \right] , \tag{23}$$

and

$$\epsilon_{2,32} = \frac{\varpi_2}{\varpi_3 \beta_{32}} \frac{1}{A_{32}} \left[Z_{23} + \frac{Z_{13}(A_{21} \rho_{21} + Z_{21})}{Z_{13} + Z_{12}} \right] . \tag{24}$$

We now have three equations of the form

$$S = \frac{\bar{J} + \epsilon \tilde{B}}{1 + \epsilon} , \tag{25}$$

where

$$\epsilon = \epsilon_1 - \beta \epsilon_2 , \tag{26}$$

and

$$\tilde{B} = \frac{\epsilon_2 (1 - \beta)}{\epsilon_1 - \beta \epsilon_2} B . \tag{27}$$

Alternatively, we can write equation (25) as

$$S - \frac{1}{\epsilon} (\bar{J} - S) = \tilde{B} . \tag{28}$$

The equation of radiative transfer establishes the dependence of the radiation field at any one point in the atmosphere on the values of S at other points. From the transfer equation (ignoring continuous absorption and emission) we can determine the coefficients W_{ij} such that

$$\bar{J}_i - S_i = \sum_{j=1}^{N} W_{ij} S_j \quad , \tag{29}$$

where N is the total number of chosen depth points. Equations (28) and (29) lead to the simultaneous equations

$$S_i - \frac{1}{\epsilon_i} \sum_{j=1}^{N} W_{ij} S_j = \tilde{B}_i \quad , \quad i = 1, 2, \cdots, N, \tag{30}$$

from which we can determine S_i given ϵ_i and $\tilde{B}_i$, $i = 1, 2, \cdots, N$. The method we use to determine the coefficients W_{ij} is similar to that developed by Athay and Skumanich (1967).

Once S is known, it follows from equations (10) and (25) that

$$\rho = \epsilon \left(\frac{\tilde{B}}{S} - 1 \right) \quad . \tag{31}$$

It also follows from equations (10) and (25) that

$$\rho = \frac{\epsilon(\tilde{B} - \bar{J})}{\bar{J} + \epsilon \tilde{B}} \quad . \tag{32}$$

We could obtain $\bar{J}$ by an integration over the above values of S, but this procedure is basically equivalent to the use of equation (29), i. e.

$$J_i = S_i + \sum_{j=1}^{N} W_{ij} S_j \quad . \tag{33}$$

If the same coefficients W_{ij} are used to obtain S and to obtain $\bar{J}$ from S, equations (31) and (32) give identical results.

4. The Unmodified Iterative Procedure

In the three-level case we begin with assumed values of ρ_{21}, ρ_{31}, and ρ_{32} at each depth; usually we set them all equal to zero. Then for each transition, ϵ and $\tilde{B}$ are calculated according to equations (26) and (27). Note that for each transition ji, ρ_{ji} does not enter the calculation of ϵ and $\tilde{B}$. Given ϵ and $\tilde{B}$, each set of simultaneous equations (30) is used to calculate S_{21}, S_{31}, and S_{32}. From the values of S, ϵ, and $\tilde{B}$ in each case, we use equation (31) to determine improved values for ρ_{21}, ρ_{31}, and ρ_{32}. This procedure is repeated until a stable solution is obtained. Often we find that after five or ten such iterations the successive values of S are identical to an accuracy of six or more significant figures.

Whenever there is a complete circuit of radiative transitions, as in Figure 2, it is possible to ask whether the solution is self-consistent. From equation (11) it follows that

$$\frac{n_i}{n_j} = \frac{\omega_i}{\omega_j}\left(\frac{a_{ji}}{S_{ji}} + 1\right) \quad , \tag{34}$$

where

$$a_{ji} = 2h\nu_{ji}^3/c^2 \quad . \tag{35}$$

In order to satisfy the condition

$$\frac{n_1}{n_2}\frac{n_2}{n_3}\frac{n_3}{n_1} = 1 \tag{36}$$

in the three-level case, we must have

$$\left(\frac{a_{21}}{S_{21}} + 1\right)\left(\frac{a_{32}}{S_{32}} + 1\right)\left(\frac{a_{31}}{S_{31}} + 1\right)^{-1} = 1 \quad . \tag{37}$$

This question of consistency does not present itself in the cases shown in Figure 1. In case (a), for example, there is no need for an independent determination of S_{32}. When $A_{32} = 0$, the term $A_{32}\rho_{32}$ drops out of our equations for the 2-1 and 3-1 transitions, and we iterate using only the functions ρ_{21}, ρ_{31}, S_{21}, and S_{31}. There are several ways in which S_{32} can be determined, but equation (37) is automatically satisfied when a stable solution is obtained.

In the general three-level problem, however, equation (37) is not necessarily satisfied, even after the values of S are not affected by further iteration. The difficulty lies in the use of equation (31) to compute ρ_{ji} at small optical depths.

5. An Alternative Calculation of ρ_{ji}

In optically thick regions of the atmosphere it is necessary to deal with the radiative rates as net rates, since pairs of single radiative rates tend to cancel one another. Formulations that disregard such cancellation do not converge properly.

In optically thin regions the use of net radiative rates can lead to difficulties. For example, if all the bound-free rates and bound-bound collision rates could be neglected compared with the net bound-bound radiative rates, equations (14) would reduce to the indeterminate form

$$\frac{n_2}{n_1} A_{21}\rho_{21} - \frac{n_3}{n_1} A_{32}\rho_{32} = 0 \quad ,$$

$$0 + \frac{n_3}{n_1}(A_{31}\rho_{31} + A_{32}\rho_{32}) = 0 \quad . \tag{38}$$

On the other hand, when equations (14) are written in terms of single rates, we have

$$\frac{n_2}{n_1}\left[A_{21}\left(1+\frac{\bar{J}_{21}}{a_{21}}\right)+Z_{21}+A_{32}\frac{\omega_3}{\omega_2}\frac{\bar{J}_{32}}{a_{32}}+Z_{23}\right]$$

$$-\frac{n_3}{n_1}\left[A_{32}\left(1+\frac{\bar{J}_{32}}{a_{32}}\right)+Z_{32}\right]=A_{21}\frac{\omega_2}{\omega_1}\frac{\bar{J}_{21}}{a_{21}}+Z_{12}\ ,$$

$$-\frac{n_2}{n_1}\left[A_{32}\frac{\omega_3}{\omega_2}\frac{\bar{J}_{32}}{a_{32}}+Z_{23}\right]+$$

$$\frac{n_3}{n_1}\left[A_{31}\left(1+\frac{\bar{J}_{31}}{a_{31}}\right)+Z_{31}+A_{32}\left(1+\frac{\bar{J}_{32}}{a_{32}}\right)+Z_{32}\right]$$

$$=A_{31}\frac{\omega_3}{\omega_1}\frac{\bar{J}_{31}}{a_{31}}+Z_{13}\ . \qquad (39)$$

Now, even when each Z_{ij} is equal to zero, we can solve these equations for n_2/n_1 and n_3/n_1, given $\bar{J}_{21}$, $\bar{J}_{31}$, and $\bar{J}_{32}$.

The basic relationship between $\bar{J}$ and S is given by equation (25), from which it follows that

$$\bar{J}=S(1+\epsilon)-\epsilon\tilde{B}\ . \qquad (40)$$

Each time we obtain values of S from the simultaneous equations (30), the corresponding $\bar{J}$ can be obtained from equation (40). We can also determine $\bar{J}$ by an integration over S, but equation (40) normally gives the same result.

Given $\bar{J}_{21}$, $\bar{J}_{31}$, and $\bar{J}_{32}$, we can compute n_2/n_1 and n_3/n_1 from equations (39), and n_3/n_2 according to

$$\frac{n_3}{n_2} = \frac{n_3}{n_1} \Big/ \frac{n_2}{n_1} \quad . \tag{41}$$

The values of S_{21}, S_{31}, and S_{32} are then obtained by equation (11). These three source functions satisfy equation (37) identically. From these corresponding values of $\bar{J}$ and S, we can determine ρ_{21}, ρ_{31}, and ρ_{32} by the defining equation

$$\rho = 1 - \frac{\bar{J}}{S} \quad . \tag{42}$$

There is a basic difference between this method of determining ρ and the direct method using

$$\rho = \epsilon \left(\frac{\tilde{B}}{S} - 1 \right) \quad . \tag{43}$$

In optically thick regions where $\rho \ll 1$, equation (42) gives almost meaningless results, except in the final iterations when $\bar{J}$ and S are nearly equal to their final consistent values. Equation (43) is far more appropriate, since we usually have either $\epsilon \ll 1$ or $S \approx \tilde{B}$.

In optically thin regions, ρ is not necessarily small compared with unity, and it is sometimes more appropriate to use equation (42) rather than equation (43) for the determination of ρ.

6. The Modified Iterative Procedure

As shown in Section 9, it is necessary to use the modified procedure in one of two alternative forms: A) In the surface region of the atmosphere where every τ_{ji} is less than τ^*, equation (42) is used to determine ρ. Otherwise, we use equation (43). B) We consider each transition separately and determine ρ by equation (42) when $\tau_{ji} < \tau^*$ and by equation (43) when $\tau_{ji} > \tau^*$.

At depth values near $\tau = \tau^*$ we use a weighted average of the two expressions for ρ. We have found that the most suitable value for τ^* is approximately 5. Sometimes the iterative corrections based on equation (42) are excessive, in which case we limit the magnitude of these corrections. Note that S in equation (42) and S in equation (43) are obtained in different ways.

The stable solution obtained in this manner is always consistent in that equation (37) is satisfied. When a stable solution is obtained, there is complete agreement between equations (42) and (43) at all depths. In some cases, as shown in Section 9, the use of equation (42) is unnecessary, and equation (43) can be used for the calculation of ρ at all depths.

In the three-level case, we often have $\tau_{32} \ll 1$ in an extended region of the atmosphere where both $\tau_{21} \gg 1$ and $\tau_{31} \gg 1$, so that S_{32} in the expression $1 - (\bar{J}_{32}/S_{32})$ could be determined from S_{21} and S_{31} by the use of equation (37). It is difficult, however, to generalize this simpler approach for cases with more than three levels. Furthermore, it is inapplicable in regions that are optically thin in all transitions.

7. On the Use of Single Radiative Rates

Since the cause of our difficulties (when they arise) apparently lies in the calculation of ρ, we should consider the determination of ϵ_1 and ϵ_2 in terms of single rates. If we retain the separate absorption and emission rates for the 3-2 transition, equations (14) have the form

$$\frac{n_2}{n_1}\left(A_{21}\rho_{21} + Z_{21} + A_{32}\frac{\omega_3}{\omega_2}\frac{\bar{J}_{32}}{a_{32}} + Z_{23}\right)$$

$$+ \frac{n_3}{n_1}\left[A_{32}\left(1 + \frac{\bar{J}_{32}}{a_{32}}\right) + Z_{32}\right] = Z_{12} \quad ,$$

$$- \frac{n_2}{n_1}\left(A_{32}\frac{\omega_3}{\omega_2}\frac{\bar{J}_{32}}{a_{32}} + Z_{23}\right)$$

$$+ \frac{n_3}{n_1}\left[A_{31}\rho_{31} + Z_{31} + A_{32}\left(1 + \frac{\bar{J}_{32}}{a_{32}}\right) + Z_{32}\right] = Z_{13} \quad . \qquad (44)$$

Equations (17) and (18) would then become

$$\epsilon^{\dagger}_{1,21} = \frac{1}{A_{21}} \left[Z_{21} + \frac{(A_{31}\rho_{31} + Z_{31}) \left(A_{32} \frac{\omega_3}{\omega_2} \frac{\bar{J}_{32}}{a_{32}} + Z_{23} \right)}{A_{31}\rho_{31} + Z_{31} + A_{32} \left(1 + \frac{\bar{J}_{32}}{a_{32}} \right) + Z_{32}} \right] \tag{45}$$

and

$$\epsilon^{\dagger}_{2,21} = \frac{\omega_1}{\omega_2 \beta_{21}} \frac{1}{A_{21}} \left\{ Z_{12} + \frac{Z_{13} \left[A_{32} \left(1 + \frac{\bar{J}_{32}}{a_{32}} \right) + Z_{32} \right]}{A_{31}\rho_{31} + Z_{31} + A_{32} \left(1 + \frac{\bar{J}_{32}}{a_{32}} \right) + Z_{32}} \right\} \tag{46}$$

In the set of simultaneous equations (30) for S_{21}, ϵ and $\tilde{B}$ would be determined from $\epsilon^{\dagger}_{1,21}$ and $\epsilon^{\dagger}_{2,21}$ rather than from $\epsilon_{1,21}$ and $\epsilon_{2,21}$. In the above equations, only the 3-2 rates have been treated as single rates. We could also treat the 3-1 rates in the same manner.

Cuny (1968) reports that such a formulation can be used successfully: given two strongly coupled transitions, the ϵ terms for one transition can be written in terms of net rates and those for the other in terms of single rates. A disadvantage of such an approach is its lack of generality. The same criticism applies to a similar method used by the author in an earlier paper (Avrett, 1966).

8. On the Use of Total Net Rates

The manner in which ϵ_1 and ϵ_2 are determined has been described briefly in Section 2. The procedure we use contains an additional provision, which is explained here.

The net radiative rate for the transition ji is given by

$$n_j(A_{ji} + \mathcal{B}_{ji}\bar{J}_{ji}) - n_i\mathcal{B}_{ij}\bar{J}_{ji} = n_jA_{ji}\rho_{ji} \quad . \tag{47}$$

The total net rate including collision terms is

$$n_j(A_{ji} + \mathcal{B}_{ji}\bar{J}_{ji} + C_{ji}) - n_i(\mathcal{B}_{ij}\bar{J}_{ji} + C_{ij})$$
$$= n_j\left[A_{ji}\rho_{ji} + C_{ji}\left(1 - \frac{b_i}{b_j}\right)\right] \quad . \tag{48}$$

Here

$$\frac{b_j}{b_i} = \left(\frac{n_j}{n_i}\right)\Big/\left(\frac{n_j^*}{n_i^*}\right) \quad , \tag{49}$$

$$C_{ij} = C_{ji}\left(\frac{n_j^*}{n_i^*}\right) \quad , \tag{50}$$

and

$$\frac{n_j^*}{n_i^*} = \frac{\omega_j}{\omega_i}e^{-h\nu_{ji}/kT} \quad . \tag{51}$$

Equations (20) and (21) for $\epsilon_{1,31}$ and $\epsilon_{2,31}$ were determined from the statistical equilibrium equations

$$\frac{n_3}{n_1}(A_{31}\rho_{31} + Z_{31} + A_{32}\rho_{32} + Z_{32}) - \frac{n_2}{n_1}Z_{23} = Z_{13} \quad , \tag{52}$$

$$-\frac{n_3}{n_1}(A_{32}\rho_{32} + Z_{32}) + \frac{n_2}{n_1}(A_{21}\rho_{21} + Z_{21} + Z_{23}) = Z_{12} \quad . \tag{53}$$

Equation (53) can be rewritten as

$$-\frac{n_3}{n_1}(A_{32}\rho_{32} + Z_{32}) + \frac{n_2}{n_1} Z_{23} = \chi_{12} \quad , \tag{54}$$

where

$$\chi_{12} = -\frac{n_2}{n_1}(A_{21}\rho_{21} + Z_{21} - \frac{n_1}{n_2} Z_{12}) \quad , \tag{55}$$

or

$$\chi_{12} = -\frac{n_2}{n_1}\left[A_{21}\rho_{21} + C_{21}\left(1 - \frac{b_1}{b_2}\right) + \mathcal{P}_{21} - \frac{n_1}{n_2}\mathcal{P}_{12}\right] \quad . \tag{56}$$

In this equation the term in brackets (multiplied by n_2) is the total 2-1 net rate, including transitions via the continuum. In the case of a two-level atom with a continuum, $\chi_{12} = 0$.

If equation (54) is used in the place of equation (53), $\epsilon_{1,31}$ and $\epsilon_{2,31}$ become

$$\epsilon_{1,31} = \frac{Z_{31}}{A_{31}} \quad , \tag{57}$$

and

$$\epsilon_{2,31} = \frac{\omega_1}{\omega_3\beta_{31}} \frac{1}{A_{31}}(Z_{13} + \chi_{12}) \quad . \tag{58}$$

These expressions are more appropriate than the original ones when χ_{12} is small — which is the case when the 2-1 transition is largely unaffected by level 3. It can be verified that these expressions follow directly from the equation for level 1, i. e., from the first of equations (12).

There are cases in which it is necessary to introduce such total net rates. However, this additional complication is not essential in the cases discussed in the following section.

9. Numerical Applications

A. A Three-Level Hydrogen Case

We now consider a simplified three-level case in which the frequencies and statistical weights correspond to the three lowest levels of atomic hydrogen. For simplicity, all bound-free transitions are ignored. Each Z_{ij} is then equal to C_{ij} rather than to $C_{ij} + \mathcal{P}_{ij}$. We assume that each C_{ij} is constant with depth, and that the temperature is constant. The absorption profiles are Gaussian with Doppler widths determined by the temperature. We choose a plane-parallel semi-infinite atmosphere, assume completely non-coherent scattering, and ignore continuous absorption and emission in every transfer equation.

The input values we select are: $\nu_{21} = 2.47 \times 10^{15}$, $\nu_{31} = 2.93 \times 10^{15}$, $\omega_1 = 2$, $\omega_2 = 8$, $\omega_3 = 18$, $A_{21} = 4.68 \times 10^{8}$, $A_{31} = 5.54 \times 10^{7}$, $A_{32} = 4.39 \times 10^{7}$, $C_{21} = C_{31} = C_{32} = 10^{5}$, and $T = 5,000°$. These data are sufficient to allow us to determine S_{21}, S_{31}, and S_{32} as functions of the optical depths τ_{21}, τ_{31}, and τ_{32}.

The unmodified procedure described in Section 4 and the modified procedure described in Section 6 were both used to solve this problem. Both methods produced the same solution, which is given in Table 1. Rapid convergence was found in each case. After ten iterations the solution changes by less than one part in 10^6. The three source functions are consistent with each other when

$$\xi \equiv \left(\frac{a_{21}}{S_{21}} + 1\right)\left(\frac{a_{32}}{S_{32}} + 1\right)\left(\frac{a_{31}}{S_{31}} + 1\right)^{-1} \tag{59}$$

is equal to unity; in the tenth iteration we find that $|\xi - 1|$ never exceeds 10^{-5} at any depth. At optical depths less than unity we find that the corresponding values of

Table 1

Solution for the three-level hydrogen problem (Case A)

τ_{21}	S_{21}/B_{21}	τ_{31}	S_{31}/B_{31}	τ_{32}	S_{32}/B_{32}
0	0.0115	0	0.00266	0	0.228
1.0(-3)	0.0116	1.60(-4)	0.00266	1.89(-14)	0.228
1.0(-2)	0.0117	1.60(-3)	0.00267	1.91(-13)	0.226
1.0(-1)	0.0127	1.60(-2)	0.00272	2.00(-12)	0.212
1	0.0193	1.60(-1)	0.00308	2.59(-11)	0.159
1.0(1)	0.0562	1.60	0.00517	6.05(-10)	0.0913
1.0(2)	0.210	1.60(1)	0.0142	2.16(-9)	0.0666
1.0(3)	0.634	1.60(2)	0.0385	7.12(-7)	0.0600
1.0(4)	0.979	1.60(3)	0.0581	1.38(-5)	0.0589
1.0(5)	1.023	1.60(4)	0.0606	1.64(-4)	0.0588
1.0(6)	1.027	1.60(5)	0.0611	1.68(-3)	0.0589
1.0(7)	1.027	1.60(6)	0.0621	1.69(-2)	0.0601
1.0(8)	1.027	1.60(7)	0.0703	1.69(-1)	0.0677
1.0(9)	1.025	1.60(8)	0.120	1.68	0.116
1.0(10)	1.019	1.60(9)	0.354	1.67(1)	0.345
1.0(11)	1.008	1.60(10)	0.816	1.65(2)	0.810
1.0(12)	1.000	1.60(11)	0.985	1.63(3)	0.985
1.0(13)	1.000	1.60(12)	0.999	1.63(4)	0.999
1.0(14)	1.000	1.60(13)	1.000	1.63(5)	1.000
B_{21} = 1.12(-11)		B_{31} = 2.26(-13)		B_{32} = 1.76(-5)	

$$\rho_{ji}^{(S)} = 1 - \frac{\bar{J}_{ji}}{S_{ji}} \tag{60}$$

and

$$\rho_{ji}^{(L)} = \epsilon_{ji}\left(\frac{\tilde{B}_{ji}}{S_{ji}} - 1\right) \tag{61}$$

are the same to within one part in 10^5. The ratios b_j/b_i are calculated in two ways: 1) from S_{ji} according to

$$\frac{b_j}{b_i} = \left[\left(\frac{\alpha_{ji}}{S_{ji}} + 1\right) e^{-h\nu_{ji}/kT}\right]^{-1} , \tag{62}$$

and 2) from

$$\frac{b_j}{b_i} = \left(\frac{n_j}{n_i}\right) \Big/ \left(\frac{n_j^*}{n_i^*}\right) , \tag{63}$$

where n_j/n_i is determined from the set of equations (39). These equations depend on the values of $\bar{J}_{21}$, $\bar{J}_{31}$, and $\bar{J}_{32}$. In the tenth iteration the corresponding values of b_j/b_i agree to an accuracy of a few parts in 10^6. These tests indicate that the solution given in Table 1 is a fully self-consistent one.

The iterative behavior of the solution by use of the unmodified procedure is indicated in Table 2. A solution accurate to better than one percent is obtained in five iterations. The iterative behavior of the modified procedure is similar. We conclude that in this case the unmodified procedure has good convergence properties, and leads to a self-consistent solution.

Table 2

Iterative behavior of S_{ji} and ξ at $\tau = 0$ for Case A

Iteration number n	$S_{21}^{(n)}/S_{21}^{(10)}$	$S_{31}^{(n)}/S_{31}^{(10)}$	$S_{32}^{(n)}/S_{32}^{(10)}$	ξ
1	0.983	19.5	0.257	0.0130
2	0.998	2.87	1.02	0.355
3	0.9994	0.992	0.892	0.900
4	0.99998	1.04	0.996	0.959
5	0.99993	0.9998	1.003	1.003
6	0.999998	1.002	0.9991	0.998
7	0.999999	1.00003	1.002	1.002
8	1.000000	0.99997	0.9998	0.99990
9	1.000000	1.00002	1.0003	1.0003
10	1	1	1	1.000002

B. The Three-Level Case with Variations in C_{ji}

In the preceding case the collisional de-excitation rates were constant throughout the atmosphere: $C_{21} = C_{31} = C_{32} = 10^5$. The case we now consider is the same except that each C_{ji} varies from 10^5 at $\tau_{21} = 1000$ to 10^3 at $\tau_{21} = 0$. The exact values we use are given in Table 3.

In this case the unmodified procedure has poor convergence properties. The results we obtain using this procedure are given in Table 4. The first column of Table 4 should be compared with the last column of Table 2. When the collision rates are constant, the unmodified method is rapidly convergent. When the collision rates vary with depth, the unmodified method is slowly convergent near the surface.

Table 3

Depth variation of the collisional de-excitation rates in Case B

τ_{21}	$C_{ji}(j > i)$
0	1.0(3)
1(-3)	1.2(3)
3(-3)	1.6(3)
1(-2)	2.5(3)
3(-2)	4.0(3)
1(-1)	6.3(3)
3(-3)	1.0(4)
1	1.6(4)
3	2.5(4)
1(1)	4.0(4)
3(1)	6.3(4)
1(2)	8.5(4)
3(2)	9.0(4)
1(3)	1.0(5)
$\tau_{21} > 1(3)$	$C_{ji} = 1(5)$

In these examples all bound-free transitions have been ignored. If the bound-free rates 1) are relatively constant with depth and 2) are much larger than the bound-bound collision rates, then each Z_{ij} is relatively constant, and the unmodified method has good convergence properties. In the standard hydrogen problem discussed by Athay et al. (1968) in these Proceedings, the bound-free rates predominate and are nearly constant with depth near the surface. This feature may explain why no convergence difficulties were encountered with the standard problem.

The last two rows in Table 4 show the results of two further iterations by use of the modified procedure, Alternative A, described in Section 6. We have solved this same case by the modified procedure starting with all $\rho_{ji} = 0$. The same final solution is obtained in about six iterations.

Table 4

Iterative behavior of ξ in Case B at various depths

Iteration number n	ξ				
	Surface	$\tau_{21} = 0.1$	$\tau_{21} = 1$	$\tau_{21} = 10$	$\tau_{21} = 100$
1	0.013	0.014	0.018	0.032	0.040
2	0.117	0.148	0.219	0.346	0.468
3	0.311	0.418	0.617	0.907	0.994
4	0.299	0.443	0.657	0.881	0.968
5	0.336	0.519	0.757	0.965	0.999
6	0.341	0.565	0.805	0.965	0.998
7	0.358	0.609	0.850	0.986	1.000
8	0.366	0.654	0.885	0.988	1.000
9	0.377	0.687	0.909	0.995	1.000
10	0.386	0.725	0.931	0.996	1.000
11*	1.008	1.008	1.006	1.001	1.000
12	1.000	1.000	1.000	1.000	1.000

*Continuation by use of the modified procedure

We conclude from this example that the unmodified procedure sometimes has poor convergence properties, and that the modified procedure is an acceptable alternative.

C. A Two-Line Problem with a Common Lower Level

We now consider a three-level case in which $A_{32} = 0$. Moreover, $A_{21} = 200$, $A_{31} = 100$, $C_{21} = C_{31} = 1$, $C_{32} = 1000$, $\omega_1 = \omega_2 = \omega_3 = 1$, and $T = 100,000°$; otherwise the input values are the same as for Case A. A solution of this problem is given by Kalkofen (1968) using a method in which the two lines are treated simultaneously. If stimulated emissions could be

neglected, this simultaneous method leads to an immediate solution without iteration. Stimulated emissions are important in this 100,000° case, and Kalkofen presents an iterative perturbation method having the property of very rapid convergence.

The discussion in this section has two purposes: 1) to add support to Kalkofen's conclusion that "standard" iterative methods for solving this type of problem are slowly convergent, and 2) to show, on the other hand, that our modified iterative method for solving this problem has acceptable convergence properties.

In Table 5 we give the iterative properties of the unmodified or standard procedure as applied to the above case. After five iterations the solution is in error by about ten percent. After ten iterations the error is about three percent.

Table 5

Iterative behavior of S_{21}, S_{31} and ξ at $\tau = 0$ for Case C by use of the unmodified procedure

Iteration number n	$S_{21}^{(n)}/S_{21}^{(exact)}$	$S_{31}^{(n)}/S_{31}^{(exact)}$	ξ
1	1.231	1.875	0.646
2	0.776	1.163	0.655
3	0.897	1.223	0.725
4	0.884	1.122	0.780
5	0.928	1.110	0.828
6	0.933	1.074	0.864
7	0.954	1.062	0.894
8	0.960	1.045	0.916
9	0.972	1.037	0.935
10	0.975	1.029	0.948
$S_{21}^{(exact)} = 0.00551$		$S_{31}^{(exact)} = 0.00755$	

When we apply the modified procedure to this case, the solution near the surface oscillates with a slowly decreasing amplitude. We damp such oscillations as follows: in iteration n we replace ρ_n by $(1 - W_\rho)\rho_n + W_\rho \rho_{n-1}$. The iterative behavior of the solutions with $W_\rho = 0.2$ and $W_\rho = 0.3$ is shown in Tables 6 and 7, respectively. In the first case the solution oscillates with a rapidly decreasing amplitude. In the second case the solution is in error by less than one percent after five iterations and by less 0.1 percent after seven iterations. The solution is given as a function of depth (in terms of b ratios) in Table 8.

Table 6

Iterative behavior of S_{21}, S_{31} and ξ at $\tau = 0$ for Case C by use of the modified procedure with $W_\rho = 0.2$

Iteration number n	$S_{21}^{(n)}/S_{21}^{(10)}$	$S_{31}^{(n)}/S_{31}^{(10)}$	ξ
1	1.232	1.874	0.646
2	1.004	1.106	0.904
3	1.053	1.015	1.036
4	0.982	1.014	0.967
5	1.015	0.998	1.015
6	0.994	1.003	0.990
7	1.006	0.999	1.006
8	0.998	1.001	0.996
9	1.003	0.999	1.002
10	1	1	0.999
$S_{21}^{(10)} = 0.005501$		$S_{31}^{(10)} = 0.007552$	

Table 7

Iterative behavior of S_{21}, S_{31} and ξ at $\tau = 0$ for Case C by use of the modified procedure with $W_\rho = 0.3$

Iteration number n	$S_{21}^{(n)}/S_{21}^{(10)}$	$S_{31}^{(n)}/S_{31}^{(10)}$	ξ
1	1.231	1.875	0.646
2	1.021	1.172	0.867
3	1.044	1.040	1.003
4	1.001	1.019	0.982
5	1.005	1.005	1.000
6	1.000	1.003	0.997
7	1.001	1.001	1.000
8	1.000	1.000	1.000
9	1.000	1.000	1.000
10	1	1	1.000
$S_{21}^{(10)} = 0.005507$		$S_{31}^{(10)} = 0.007550$	

Table 8

Solution for Case C

τ_{21}	b_2/b_1	b_3/b_1
0	0.079	0.081
1(-2)	0.080	0.082
3(-2)	0.082	0.084
1(-1)	0.086	0.088
3(-1)	0.095	0.096
1	0.118	0.119
3	0.166	0.166
1(1)	0.272	0.272
3(1)	0.435	0.435
1(2)	0.681	0.681
3(2)	0.873	0.873
1(3)	0.969	0.969
3(3)	0.992	0.992
1(4)	0.998	0.998
3(4)	0.999	0.999
1(5)	1.000	1.000

D. A Two-Line Problem with a Common Upper Level

To illustrate the generality of the method described here, we now invert the atomic configuration of Case C. Then $A_{21} = 0$, $A_{31} = 100$, $A_{32} = 200$, $C_{31} = C_{32} = 1$, $C_{21} = 1000$, and $\nu_{21} = 0.46 \times 10^{15}$; otherwise the input values are the same as in Case C. The two radiative transitions now have a common upper level.

This case and the preceding one have similar properties. The iterative behavior with $W_\rho = 0.3$ is shown in Table 9. The solution is given in Table 10. A comparison of Tables 8 and 10 shows that the two lines are more strongly coupled together in the case of a common upper level.

Table 9

Iterative behavior of S_{31}, S_{32} and ξ at $\tau = 0$ for Case D by use of the modified procedure with $W_\rho = 0.3$

Iteration number n	$S_{31}^{(n)}/S_{31}^{(10)}$	$S_{32}^{(n)}/S_{32}^{(10)}$	ξ
1	1.213	1.845	1.492
2	1.195	0.999	0.839
3	0.995	1.081	1.086
4	1.022	0.995	0.972
5	0.997	1.012	1.014
6	1.002	0.998	0.995
7	0.999	1.001	1.002
8	1.000	1.000	0.999
9	1.000	1.000	1.000
10	1	1	1.000
$S_{31}^{(10)} = 0.00760$		$S_{32}^{(10)} = 0.00571$	

Table 10

Solution for Case D

τ_{21}	b_3/b_1	b_3/b_2
0	0.08202	0.08204
1(-2)	0.083	0.083
3(-2)	0.085	0.085
1(-1)	0.090	0.090
3(-1)	0.100	0.100
1	0.129	0.129
3	0.190	0.190
1(1)	0.323	0.323
3(1)	0.517	0.517
1(2)	0.763	0.763
3(2)	0.916	0.916
1(3)	0.979	0.979
3(3)	0.994	0.994
1(4)	0.998	0.998
3(4)	1.000	1.000
1(5)	1.000	1.000

E. A Four-Level Sodium Case

The first four-level case we attempted is the one indicated in Figure 3, consisting of the sodium D_1 and D_2 lines together with the 8183 Å and 8195 Å subordinate lines. This problem turns out to be an exceptionally difficult one. In the course of a lengthy investigation we tried many different methods without success. The only method we have found to work is the one described in Section 6, using Alternative B. We now define a particular problem and give its solution.

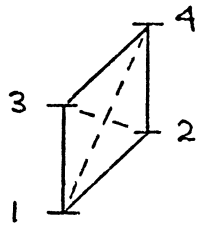


Figure 3

We use the input values $\nu_{21} = 0.509 \times 10^{15}$, $\nu_{31} = 0.510 \times 10^{15}$, $\nu_{41} = 0.875 \times 10^{15}$, $\omega_1 = 2$, $\omega_2 = 2$, $\omega_3 = 4$, $\omega_4 = 4$, $A_{21} = 6.05 \times 10^{7}$, $A_{31} = 6.05 \times 10^{7}$, $A_{42} = 4.17 \times 10^{7}$, $A_{43} = 8.34 \times 10^{7}$, $C_{21} = C_{31} = C_{41} = 10^{4}$, $C_{42} = 5 \times 10^{4}$, $C_{43} = 10^{5}$, $C_{32} = 10^{6}$, and $T = 5000°$. The same simplifications imposed in the previous section are imposed here. The solution we obtain is a fully consistent one and is given in Table 11.

Here the collision rates are constant with depth, and it is possible to obtain this solution if we use the unmodified method. However, about 40 iterations are required. When the collision rates vary with depth, the unmodified method does not yield a consistent solution.

In Table 12 we show the initial iterative behavior of the solution when we decrease the values of C_{ji}, $j > i$, by the factor 10 in the region $0 \leq \tau_{31} \leq 1$. The quantity ξ' is given by

$$\xi' = \left(\frac{a_{32}}{S_{32}} + 1\right)\left(\frac{a_{43}}{S_{43}} + 1\right)\cdot\left(\frac{a_{42}}{S_{42}} + 1\right)^{-1} , \qquad (64)$$

in analogy with equation (59). (S_{32} is obtained from the full set of statistical equilibrium equations.) Both ξ and ξ' are equal to unity when the computed population ratios are consistent with each other.

Table 11

Four-level sodium solution

τ_{31}	S_{31}/B_{31}	τ_{21}	S_{21}/B_{21}	τ_{43}	S_{43}/B_{43}	τ_{42}	S_{42}/B_{42}
0	0.00986	0	0.0133	0	0.0418	0	0.0311
1.02(-3)	0.00988	5.12(-4)	0.0133	2.84(-7)	0.0418	1.92(-7)	0.0311
1.02(-2)	0.0100	5.12(-3)	0.0134	2.86(-6)	0.0417	1.93(-6)	0.0312
1.02(-1)	0.0109	5.12(-2)	0.0141	3.00(-5)	0.0405	1.98(-5)	0.0314
1.02	0.0165	5.12(-1)	0.0186	3.90(-4)	0.0362	2.34(-4)	0.0321
1.02(1)	0.0470	5.12	0.0453	9.04(-3)	0.0316	4.54(-3)	0.0328
1.02(2)	0.171	5.11(1)	0.168	3.12(-1)	0.0367	1.53(-1)	0.0374
1.01(3)	0.531	5.10(2)	0.530	1.05(1)	0.123	5.22	0.123
1.01(4)	0.925	5.09(3)	0.925	2.19(2)	0.577	1.10(2)	0.578
1.01(5)	0.995	5.08(4)	0.995	2.68(3)	0.955	1.34(3)	0.958
1.01(6)	1.000	5.08(5)	1.000	2.76(4)	11.000	1.38(4)	1.000
B_{31} = 1.47(-5)		B_{21} = 1.48(-5)		B_{43} = 2.22(-5)		B_{42} = 2.22(-5)	

Table 12

Iterative behavior of ξ and ξ' at various depths in Case E by use of the unmodified procedure

Iteration number n	ξ				
	Surface	$\tau_{31} = 0.03$	$\tau_{31} = 0.8$	$\tau_{31} = 5$	$\tau_{31} = 50$
1	1.41	1.39	1.20	1.01	0.95
2	1.58	1.55	1.35	1.12	1.01
3	1.22	1.22	1.15	1.05	0.99
4	1.14	1.14	1.11	1.05	1.00
5	1.08	1.08	1.07	1.03	1.00
6	1.04	1.04	1.04	1.03	1.00
Iteration number n	ξ'				
	Surface	$\tau_{31} = 0.03$	$\tau_{31} = 0.8$	$\tau_{31} = 5$	$\tau_{31} = 50$
1	1.29	1.30	1.32	1.37	1.62
2	0.98	1.03	1.08	1.01	0.95
3	0.67	0.71	0.80	0.82	0.83
4	0.64	0.70	0.82	0.86	0.85
5	0.60	0.67	0.80	0.84	0.83
6	0.59	0.66	0.80	0.85	0.84

In Table 12 the values of ξ' near the surface tend to converge upon values that differ from unity. Subsequent iterations are of little help. However, the modified procedure yields a solution with $\xi = \xi' = 1$ at all depths.

F. A Five-Level Calcium Case

Finally, we consider the five-level CaII configuration shown in Figure 4. We use the following input values: $\nu_{21} = 0.410 \times 10^{15}$, $\nu_{31} = 0.412 \times 10^{15}$, $\nu_{41} = 0.756 \times 10^{15}$, $\nu_{51} = 0.763 \times 10^{15}$, $\omega_1 = 2$, $\omega_2 = 4$, $\omega_3 = 6$, $\omega_4 = 2$, $\omega_5 = 4$, $A_{51} = 1.4 \times 10^{8}$, $A_{41} = 1.4 \times 10^{8}$, $A_{53} = 7.2 \times 10^{6}$, $A_{52} = 8.1 \times 10^{5}$, $A_{42} = 7.8 \times 10^{6}$, $C_{51} = 5.1 \times 10^{4}$, $C_{41} = 5.1 \times 10^{4}$, $C_{53} = 1.4 \times 10^{5}$, $C_{52} = 1.6 \times 10^{4}$, $C_{42} = 1.6 \times 10^{5}$, $C_{31} = 8.2 \times 10^{3}$, $C_{21} = 8.2 \times 10^{3}$, $C_{54} = 4.8 \times 10^{6}$, $C_{32} = 1.0 \times 10^{7}$, $C_{43} = 1.0 \times 10^{3}$, and $T = 5000°$. These collision rates have been obtained from the data compiled by Linsky (1968) (and correspond to an electron density of roughly 10^{11}). The consistent solution we obtain is given in Table 13. This case is not quite as severe as the four-level sodium case, but has the same basic characteristics.

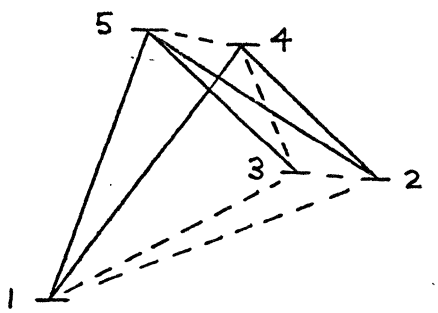


Figure 4

10. Final Remarks

The conclusions reached in the previous section are summarized at the end of Section 2. Our method of solution is outlined in Sections 3 through 6. A more complete discussion is given by Avrett and Loeser (1968a, b, c) in papers dealing with a) the determination of $\bar{J}$ from S by means of the equation of radiative transfer [cf. eq. (29)], b) general methods for handling the equations of statistical equilibrium and the coupling between different transitions, and c) a description of our general computer program PANDORA,

Table 13

Five-level calcium solution

τ_{41}	S_{41}/B_{41}	τ_{51}	S_{51}/B_{51}	τ_{42}	S_{42}/B_{42}	τ_{52}	S_{52}/B_{52}	τ_{53}	S_{53}/B_{53}
0	0.0247	0	0.0207	0	0.0635	0	0.0534	0	0.0534
1.04(-3)	0.0248	2.03(-3)	0.0208	4.44(-6)	0.0636	8.69(-7)	0.0535	7.71(-6)	0.0535
1.04(-2)	0.0251	2.03(-2)	0.0212	4.45(-5)	0.0640	8.72(-6)	0.0544	7.73(-5)	0.0544
1.04(-1)	0.0274	2.03(-1)	0.0243	4.55(-4)	0.0672	8.90(-5)	0.0598	7.89(-4)	0.0598
1.04	0.0415	2.02	0.0421	5.10(-3)	0.0839	9.97(-4)	0.0853	8.85(-3)	0.0853
1.02(1)	0.107	1.98(1)	0.110	7.38(-2)	0.132	1.44(-2)	0.136	1.28(-1)	0.136
9.91(1)	0.256	1.93(2)	0.261	1.06	0.239	2.07(-1)	0.244	1.83	0.244
9.82(2)	0.601	1.91(3)	0.601	1.13(1)	0.571	2.22	0.573	1.97(1)	0.573
983(3)	0.943	1.91(4)	0.943	1.10(2)	0.937	2.16(1)	0.937	1.91(2)	0.937
9.84(4)	0.996	1.91(5)	0.996	1.08(3)	0.993	2.12(2)	0.993	1.88(3)	0.993
9.84(5)	1.000	1.91(6)	1.000	1.08(4)	1.000	2.11(3)	1.000	1.87(4)	1.000
B_{41} = 4.49(-6)		B_{51} = 4.32(-6)		B_{42} = 2.29(-5)		B_{52} = 2.27(-5)		B_{53} = 2.27(-5)	

together with applications to "realistic" atmospheres. In addition, we give a more complete list of references to other work.

I am indebted to Yvette Cuny and Wolfgang Kalkofen for a number of critical discussions, and to Rudolf Loeser for his assistance and collaboration in all of the research reported here.

References

Athay, R. G., Avrett, E. H., Beebe, H. A., Johnson, H. R., Poland, A. I. and Cuny, Y. 1968 (these Proceedings).

Avrett, E. H. 1966, Ap. J., 144, 59.

Avrett, E H. and Loeser, R. 1968a, b, c, Smithsonian Special Reports (in final preparation).

Cuny, Y. 1968 (these Proceedings).

Kalkofen, W. 1968 (these Proceedings).

Linsky, J. L. 1968, Smithsonian Special Report No. 274 (vol. 2).